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Quantum Electrodynamics of Multiphoton Processes: Laser Physics, Thermal Radiation and Astrophysics

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Abstract QED description of multiphoton processes (MPP) requires special account of densities $j(\omega)$ of incident photons fluxes. The reaction rate of N-photon process in laser field is expressed via $(j/j_0)^N$, where $j_0 = 1/\sigma(\omega)\tau(\omega)$ is the density of reactions saturation and/or the threshold of opening new channel of reaction, $\sigma(\omega)$ and $\tau(\omega)$ are the cross-section of elastic scattering and its duration. The calculations lead to the known plateau in spectra of higher harmonics generations. Processes of multiphoton ionization require also determination of the duration of momentum accumulation $\tau(\mathbf{p})$ and reaction rate depends on its conformity with $\tau(N\omega)$. The method is generalized onto temperature fields. It allows to consider phase transition of the first kind as radiative transition between two levels (gas and condensate) and to determine their thresholds as the function of temperature and latent heat; it predicts a characteristic emission of allocated latent heat. The second rich field for the MPP theory is astrophysics, in which they can explain some singularities of spectra of stars and nebulae bursts: disappearance or suppression of low-frequency parts of spectra via energy pumping into higher frequencies, narrowing of lines, etc. MPP of higher harmonics generation and pairs birth are possible also on vacuum loops.

Keywords Multiphoton · QED · Thermal radiation · Astrophysical spectra

1 Introduction

Quantum electrodynamics (QED) examines, as a rule, processes with two particles in an initial state. From here, apparently, had been appeared a conviction that processes with big number of initial particles are the following terms of a general decomposition of S-matrix over parameter of interaction $\alpha = 1/137$. Thus, a direct calculation, for example, of the

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higher harmonics generation with matrix element containing α^N, $N \gg 1$, is seemed senseless. Probably therefore multiphoton processes (MPPs) are examined as the extremely nonlinear effects that are demanding the description by averaged functions such as nonlinear susceptibilities (e.g., the general reviews with the detailed bibliography: Brabec and Krauz [7], Dell'Anno et al. [11]).

Meanwhile for the description, for example, of process $N\gamma(\omega) + e \to e + \gamma(N\omega)$ the matrix element of $(N+1)$-order is the basic one, it is not a N-term of S-matrix perturbative decomposition (perturbative decompositions lead to appearance of additional internal lines in the Feynman graphs only). With taken into account the fluxes density of incident particles, i.e. at the correct transition to limits of the quantization volumes, the smallness of constants of interaction is compensated by other factors [33, 37].

The analysis of calculated reaction rates results in automatic occurrence of factors of duration of elementary processes. Thus, as against many known constructions (e.g., Muga et al. [30]) durations of interaction are not entered ad hoc, artificially: they initially are present in QED (the general theory of the durations of interaction: Perel'man [38, 39]). Just the existence of these factors give physical sense of the QED description of MPPs.

All this shows an opportunity of the primary consideration of MPPs in the frameworks of QED with the strict observance of all its instructions. The analysis of received results can be carried out the most simply and physically evidently, in our opinion, with taking into account the conception of duration of interaction, which should be considered as the component of general field theory.

Further such simplest temporal functions will be used: temporal functions are introduced in general but nonrelativistic form as

$$\tau(\omega) \equiv \tau_1 + i\tau_2 = (\partial/i\partial\omega)\ln S(\omega), \tag{1.1}$$

where $S(\omega) = |S(\omega)|\exp(i\Phi(\omega))$ is the response function (matrix element, propagator) of investigated process. In this expression the Wigner-Smith function,

$$\tau_1 = \operatorname{Re}\tau(\omega) = \partial\Phi/\partial\omega, \tag{1.2}$$

describes the delay duration at the scattering process via the variation of response phase. The function

$$\tau_2 = \operatorname{Im}\tau(\omega) = (\partial/\partial\omega)\ln|S(\omega)| \tag{1.3}$$

corresponds to the duration of formation of outgoing photon (its "dressing" or "redress") via the amplitude alteration.

These functions can be considered as the self-functions of the reciprocal Schrödinger equation [39]:

$$\frac{\partial}{i\partial\omega}S = \tau(\omega)S, \tag{1.4}$$

i.e. the temporal functions corresponds to the operator $\partial/i\partial\omega$ that can be generalized as the covariant operator $\partial/i\partial p_\mu$, canonically conjugated to the energy-momentum operator $i\partial/\partial x_\mu$. By the Ward-Takahashi identity of QED the covariant temporal functions can be expressed through the particle Green function $G(p)$ and the vertex part $\Gamma_\mu(p,q;p-q)$ as

$$\xi_\mu(p) \equiv \partial\ln G/i\partial p_\mu = \frac{1}{2}\{G(p)\Gamma_\mu(p,p;0) + \Gamma_\mu(p,p;0)G(p)\}. \tag{1.5}$$

It shows that the temporal measurement corresponds to addition of zero-energy line to electron line of corresponding Feynman graph (it seems equivalent to the "Larmor clock" frequently used for such descriptions [30]). This general expression will be needed only for the overview of the QED approach to the theory of MPPs in the Sect. 2.

The Sect. 3 is devoted to the detailed calculations of stimulated radiation in two-level systems. These calculations allow to revealing the most general sense of MPP in the scope of QED, including an appearance of singularities of reaction rates and thresholds of new processes as functions of flux powers. In the Sects. 4 and 5 the main MPPs on single atoms and free electrons, the higher harmonics generation (HHG) and multiphoton ionization (MPI) with acceleration of free electrons, are considered.

In the Sect. 6 the stimulated radiation in the thermal field is examined. On the one hand it is the direct continuation of Sect. 3, but this method can be directly applied to phase transitions of the first kind if two phases (e.g. gas and liquid) can be considered as "electronic levels". Such consideration leads to prediction of characteristic radiation of latent heat [34]. Some effects of this radiation were observed [38, 41]. These processes can explain the known IR excess of some nebulae and young stars spectra [35].

The Sect. 7 is devoted to astrophysical applications of the theory of MPPs. The most bright objects must be characterized by such photons density of lower frequencies part (in visual range and so on) that the MPP should have inevitable consequences in them. Therefore astrophysics would be the second richest field for the theory of MPPs.

The results and some perspectives are summed in the Conclusions.

2 Overview of General Theory

Let's consider interaction of n photons with electron, free or bound:

$$\begin{aligned}\gamma(\omega_1,\mathbf{k}_1)+\cdots+\gamma(\omega_n,\mathbf{k}_n)+e(E,\mathbf{p})\\ \longrightarrow e(E',\mathbf{p}')+\gamma(\omega_1',\mathbf{k}_1')+\cdots+\gamma(\omega_m',\mathbf{k}_m').\end{aligned} \tag{2.1}$$

In the lowest order this process is described by the standard matrix element:

$$S_{n+m}=\frac{(-e)^{n+m}}{(n+m)!}\int dx_1\cdots dx_{n+m}\widehat{T}\big\{N(\overline{\Psi}\widehat{A}\Psi)_1\cdot\cdots\cdot N(\overline{\Psi}\widehat{A}\Psi)_{n+m}\big\}. \tag{2.2}$$

For processes on a bound (atomic) electron its propagator in the Furry representation can be taken in the Low form [28] as

$$G(x_1,x_2)\approx\frac{1}{2\pi i}\sum_n\psi_n(\mathbf{r}_1)\overline{\psi}_n(\mathbf{r}_2)\int_{-\infty}^{\infty}\frac{\exp[i\omega(t_1-t_2]}{E_n+\omega-i\Gamma_n/2}d\omega, \tag{2.3}$$

external electromagnetic field can be quantized in the volume V in plane wave representation:

$$\widehat{A}_{i,f}=\sum_{\mathbf{k}}\frac{\widehat{e}a^{\pm}}{\sqrt{2\omega_{i,f}V}}\exp\{\pm i(\mathbf{k}_{i,f}\mathbf{R}-\omega_{i,f}t)\}, \tag{2.4}$$

where R is the radius-vector of the center of atom.

After integration over all time variables

$$S_{n+m}=2\pi Q_{n+m}V^{-(n+m)/2}\delta\Big(\sum\omega_n+E_0-\sum\omega_m'-E_f\Big), \tag{2.5}$$

where

$$Q_{n+m}=i(ie)^{n+m}\frac{C_{n,m}}{(n+m)!}\frac{\langle f|\widehat{e}\exp i\mathbf{k}'_m\mathbf{R}|n+m-1\rangle}{\sqrt{2\omega_{n+m}}}$$
$$\times\prod_{q=1}^{n+m-1}\sum_{l\neq\lambda}\frac{\langle l|\widehat{e}\exp i\mathbf{k}'_m\mathbf{R}|\lambda\rangle}{\sqrt{2\omega_q}(E_l-E_0-\sum_1^q\omega_k-i\Gamma_q/2)},\tag{2.6}$$

at photons absorption and emission $\omega_q\gtrless 0$, the standard designations of partial matrix elements are used, the factorials corresponding to permutations of identical particles are grouped in the $C_{n,m}$.

The reaction rate is determined as

$$dR_{n+m}=\lim_{V\to\infty}2\pi\overline{|Q_{n+m}|^2}\frac{dN_1}{V}\cdot\dots\cdot\frac{dN_n}{V},\tag{2.7}$$

dN_i is the number of incident photons of type i in the energy interval $(\omega_i,d\omega_i)$, the matrix element is averaged over polarization of initial photons and summed over polarizations of outlet photons interacting with single electron:

$$\overline{|Q_{n+m}|^2}=\frac{2j_f+1}{2^{n+m}(2j_i+1)}|Q_{n+m}(\widehat{e}\to\gamma_\nu)|^2.\tag{2.8}$$

At calculation of processes with two-particle initial states, it is possible to omit V in (2.7) by the known procedures, however in general case the situation is more complicated. The transition $V\to\infty$ is performed for outlet and incident photons, respectively, as

$$\frac{1}{V}\to\frac{d\mathbf{k}_f}{(2\pi)^3},\qquad\frac{dN_i}{V}\to f(t,\mathbf{r}|\omega,\mathbf{k})d\mathbf{k}_i.\tag{2.9}$$

For unidirectional monochromatic laser flux of photons density ρ the last distribution is simplified,

$$f(t,\mathbf{r}|\omega,\mathbf{k})\to\rho\delta(\mathbf{k}-\mathbf{k}_0),\tag{2.10}$$

but can be generalized for the case of some fluxes with different frequencies and directions, in thermal field it would be expressed via the Planck distribution and so on.

If the interference between levels in (2.6) can be neglected, all integrations over angles in (2.8), except transitions into the continuous spectrum, can be executed:

$$\int d\Omega\left|\left\langle l\left|\gamma_\mu e^{i\mathbf{kR}}\right|\lambda\right\rangle\right|^2\to 8\pi^2\Gamma_l/e^2\omega_{l\lambda}.\tag{2.11}$$

The integration over frequencies can be executed by (2.10) and/or by the Lorentz profiles in the square of expression (2.6).

For the reactions in the continuous flux, where number of interacting photons is not fixed, the complete reaction rate can be determined, after all integrations, as

$$R(j)=\sum_{n,m=0}^{\infty}R_{n,m}\to\sum_0^\infty a_n\rho^n,\tag{2.12}$$

i.e. as the virial decomposition over the photons density $\rho=j/c$ in vacuum.

At calculation of (2.12) the question of convergence of these series appears. The radius of convergence determines saturation of corresponding process and an opportunity of opening of new channels of reactions. Notice that presence of thresholds on number of particles is the feature of the theory of MPPs; in nuclear physics, for example, thresholds are determined by the energy of single incident particle.

Some basic features of MPPs can be revealed at consideration of such example.

Let's consider the most usual MPP of type

$$\gamma(\omega_1) + \cdots + \gamma(\omega_n) + A \longrightarrow B + C. \tag{2.13}$$

As the main graph of the process is one-connected, its matrix element

$$S_{fi}^{(n)} = M \prod_{q=1}^{n} \left\{ G(p_q)\Gamma_\mu(p_q, p_q + k)A_\mu(k) \right\} \Phi_{in}(p_0), \tag{2.14}$$

where $\Phi_{in}(p_0)$ is the wave function of initial state of scatterer, bound or free, M is the matrix element of decay of excited intermediate state into the final state: $A^* \to B + C$.

In the low frequencies limit ($\hbar\omega/mc^2 \ll 1$, $\lambda/r_0 \gg 1$) it can be taken in the accordance with (1.4) that

$$\Gamma_\mu(p, p + k)G(p) \rightsquigarrow \Gamma_\mu(p, p)G(p) = iG^{-1}(p)\xi_\mu(p), \tag{2.15}$$

i.e. the temporal function can be naturally introduced. In our limit $\xi_\mu(p) \to \tau(\omega)$. Hence each absorbed photon introduces into reaction rate the factor:

$$\frac{4\pi e^2}{2\omega V} \left\{ e_\mu \Gamma_\mu(p, p + k)G(p)e_\nu \right\} \xi_\nu(q\omega) \approx \frac{4\pi}{2\omega V} M'\tau, \tag{2.16}$$

where M' corresponds to the matrix element of elastic e-γ scattering and in the considered limit for it the classical Tompson amplitude $r_0 = e^2/mc^2$ can be taken. For nonpolarized photons, in accordance with the optical theorem of QED, $4\pi c r_0/2\omega = \sigma_{tot}$.

It shows that the reaction rate represents the series over

$$j/j_0(\omega) \equiv j\sigma_{tot}\tau_2(\omega), \tag{2.17}$$

and therefore clears up the physical sense of (2.12): the magnitude of $j_0 = 1/\sigma_{tot}\tau_2$ represents the saturation value of photon flux.

Thus suggested theory can be characterized as the S-matrix decomposition over the parameter (j/j_0) and can be comparing with other ones.

In the early period of MPPs investigations the processes of MPI were modeled in the quasiclassic theory by consideration of electron liberation from the potential well of depth I in the field $\mathbf{E} = \mathbf{E}_o \cos\omega t$. These calculations are leading to the Keldysh adiabatic parameter [22]:

$$\xi_o^2 = 2m\omega^2|I|/e^2\mathbf{E}_0^2. \tag{2.18}$$

At the further investigations (e.g. [43], and references therein) this parameter was slightly generalized by including the complete absorbed energy $\Sigma = N\hbar\omega$:

$$\xi_o^2 = 2m\omega\Sigma/e^2\mathbf{E}^2 \to J/U_p, \tag{2.19}$$

where U_p is the ponderomotive potential.

The introduction of a free electron's intensity parameter [42] as

$$Z = 2U_p/mc^2 \to (\hbar/2mc^2)\sigma j \tag{2.20}$$

supposes that the duration of process is determined by the Compton time $\tau_C = \hbar/mc^2$.

It must be concluded that none of these parameters take into account the momentum acquisition at MPI. Therefore, they can be suitable, in principle, for the HHG examination, but not to distinguishing between HHG and MPI at all. May be added that these parameters do not contain the characteristics of scatterers and it complicates the consideration of such phenomenon as atomic stabilization. (Some other parameters are considered by Perel'man and Arutynian [40].)

Let's define for further examinations the conditions at which the atom can be considered as an isolated object relative to MPP. The volume, in which occurs single non-resonant e-γ interaction, can be determined as

$$V_{int} = \sigma_T c\tau_2 \to (4\pi c^2/\omega^2)r_0. \tag{2.21}$$

Process can be examined as happening on the isolated atom, if the density ρ_a [atoms/cm^3] of considered substance (or, more precisely, the density of upper electrons) is those, that

$$\rho_a V_{int} \le 1. \tag{2.22}$$

This condition can be rewritten in a more simply and physically evident form through the plasma frequency as

$$\omega \ge \omega_p. \tag{2.23}$$

Thus, the execution of MPPs should go faster than excitation of plasma waves and transferring of energy into collective oscillations.

3 Stimulated Radiation

The notion of stimulated (induced) radiation appears at consideration of resonant processes:

$$\gamma(\omega_0) + e_B^*(E_1) \to e_B(E_0) + 2\gamma(\omega_0) \tag{3.1}$$

at $\omega_0 = E_1 - E_0$.

Its probability is defined in the resonance field of intensity $J = \hbar\omega j(\omega)$ by the Einstein coefficients [12] as

$$W = A + Bj(\omega_0). \tag{3.2}$$

The linear dependence on photons intensity can be generalized onto more complicated processes (cf. Einstein and Ehrenfest [13]). So, the consideration of processes in an intensive monochromatic field requires an account of possibilities of additional absorption-reemission of photons:

$$N\gamma(\omega_0) + e_B^* \to e_B + (N+1)\gamma(\omega_0). \tag{3.3}$$

Its probability must be examined in a more general form comparable with (2.13):

$$W = A + \sum_n B_n(\omega, \omega_0, \Gamma) j^n(\omega), \tag{3.4}$$

i.e. in the form of "virial" series with B_n as functions of parameters of caterers considered for simplicity as two-level systems.

Such representation, if it would be summed, can lead to the radius of convergence for processes of stimulated radiation, to their saturation and to thresholds of another processes. Let's consider its deduction in some details.

The amplitude of transition (3.3) can be written in the form of clusters decomposition as

$$\begin{aligned}&\left\langle (N+1)\gamma, e\,|S|\,N\gamma, e^*\right\rangle\\ &\quad = 2\pi i \prod_1^N \frac{n!(n+1)!}{(2n+1)!}\left\langle (n+1)\gamma, e\,|T|\,n\gamma, e^*\right\rangle \delta\Big(\sum E_f - \sum E_i\Big) Q_n,\end{aligned} \tag{3.5}$$

where Q_n corresponds to the density of states, initial and final, i.e. is determined via densities of incident photons. By taken into account its further squaring

$$Q_n^2 = \prod_1^n \frac{\rho_i(\mathbf{k})d\mathbf{k}_i}{2\omega_i}\prod_1^{n+1}\frac{d\mathbf{k}_f}{2\omega_f(2\pi)^3}, \tag{3.6}$$

where $\rho_i(\mathbf{k})$ is the density of photons of type i, in the monochromatic coherent flux $\rho_i(\mathbf{k}) = \rho\delta(\mathbf{k}-\mathbf{k}_0)$.

At $|\omega-\omega_0| < \Gamma/2$ all diagrams of (3.5) with nonresonant factors instead of resonant ones and all S-matrix terms with A_μ^2 in the Hamiltonian can be omitted. With these approximations the reaction rate with $N=1$ gives

$$R_{2,1} = \frac{64\pi}{9\omega^2} j\Gamma \frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2/4}. \tag{3.7}$$

For $n \gg 1$ with only resonant terms taken into account the partial reaction rate

$$R_{n+1,n} \approx \pi\Gamma n(j/j_0)^n \tag{3.7$'$}$$

and the complete reaction rate is summed at $j < j_0$ via the higher hypergeometric function:

$$R = \sum_0^\infty R_{n+1,n} = {}_3F_2(2,2,1;3/2,3/2;j/j_0) \approx \pi\Gamma\frac{j/j_0}{(1-\ j/j_0)^2}, \tag{3.8}$$

where the designation

$$j_0 = \frac{\omega^2}{\pi c^2}\frac{(\omega-\omega_0)^2+\Gamma^2/4}{\Gamma} \to 1/\sigma_{el}\tau_1 \tag{3.9}$$

is introduced. (To simplify the writing, here and below we omit factors arising from averaging over the initial polarizations and summing over the final polarizations.)

The expression of j_0 via the cross-section of elastic scattering σ_{el} and the duration of time delay at this process τ_1 (cf. (1.2)) evidently visualizes the physical sense of the nonremovable pole of reaction rate: this flux density corresponds to the saturation of induced radiation and, as can be demonstrated, the possibility of opening of new channel, the higher harmonics generation.

The most interesting physical result consists here in the natural appearance of temporal characteristic of scattering process, which is usually introduced in the theory ad hoc, by hands only.

Thus the parameter of multiphoton reaction rates decomposition must be of order (j/j_0), i.e. must contain temporal characteristics of interactions (further decomposition of each partial matrix element goes, naturally, over degrees of $e^2/\hbar c$). These results are deduced in [33], by another method also, there is considered also stimulated radiation at irradiation in restricted time and so on.

Threshold of the beginning of induced radiation at the strictly resonant irradiation of excited gas is enough low. So, at $\omega_0 = 10^{15}$ and $\Gamma = 10^8$ it is received that

$$I_{\min} = \hbar\omega_0 j(\omega_0) = \hbar\omega_0^2\Gamma/4\pi c^2 \sim 10^{-3}\ [\mathrm{W/cm^2}]. \tag{3.10}$$

Therefore these phenomena could be in part investigated up to the invention of lasers.

Our consideration, certainly, is much idealized: it is assumed that values of j are independent of time, i.e. their fluctuations are small on time intervals $\tau_2 \sim 1/\omega_0$, when the electron can be considered as virtual. If the distance between inlet photons is more $c\tau_2$, the matrix element is splitted onto product of independent multipliers.

The formal requirement to uniformity and intensity of radiation flux can be expressed as the condition:

$$\int_0^{\tau_2} j(t,\omega)dt \geq 1/\sigma(\omega). \tag{3.11}$$

Now we shall consider the parameters of levels, their widths and shifts in the field. In the asymptotically weak field these magnitudes are described, in accordance with Low [28], as

$$\delta\omega_n - \frac{i}{2}\Gamma_n = \frac{e^2}{16\pi^2}\int\frac{d\mathbf{k}}{k}\sum_m \frac{\left|\left\langle n\left|\gamma_\mu \exp(i\mathbf{kR})\right|m\right\rangle\right|}{\omega_n^{(0)} - \omega_m^{(0)}(1-i0) - k\,\mathrm{sgn}\,\omega_m^{(0)}}. \tag{3.12}$$

External field can be taken into account by the substitution: $k_\mu \to k_\mu - ieA_\mu$, at which the nominator of (3.12) receives additive terms that in the dipole approximation are of the type of $\Omega_{n,m} = \langle n\,|\mathbf{dE}|\,m\rangle$ and its square. Therefore the widths of levels and their shifts become to

$$\begin{aligned}\Gamma_n &= \Gamma_n^{(0)} + 2\sum_m \Omega_{n,m} + a\sum_m \Omega_{n,m}^2/\omega + \cdots,\\ \delta\omega_n &= \delta\omega_n^{(0)} + \alpha j^{1/2} + \beta j + \cdots.\end{aligned} \tag{3.13}$$

Note that in the high frequencies field (3.13) correspond to so-called theorem of squares:

$$\Gamma t \to 2\int_0^t (\mathbf{dE})_{21}dt \tag{3.14}$$

that naturally leads to $\Delta\Gamma \sim j^{1/2}$.

In the two-level system $\Gamma^{(0)} \ll \Omega \ll \omega$ in an intensive field usually and therefore the half-width of level can be replaced by the Raby frequency $\Omega = (\mathbf{dE})_{21}$. In a more general case it can be taken that $\Gamma \to 2\,\langle n\,|\mathbf{dE}|\,m\rangle$ and near to the resonance ($\Gamma > |\Delta\omega|$) such substitution is possible:

$$\tau_1 = \frac{\Gamma/2}{\Delta\omega^2 + \Gamma^2/4} \to \frac{1}{\sqrt{\Delta\omega^2 + \Omega_{nm}^2}}. \tag{3.15}$$

It shows that atomic electrons in the intensive field must be in continuous beatings with frequency $1/\tau_1$ between levels, i.e. in the sequential absorption-reemission processes. This result completely coincides with the quantum mechanical calculation [26]. But our consideration shows that such beatings take place at $j \leq j_0$ only, higher fields must be examined additionally.

Let us consider now the dependence of level parameters on flux intensity via the substitution of (3.9) into (3.8) that can be rewritten at $j < j_0$ in the Lorentz form:

$$R \approx Aj/\left[(\Delta\omega - \delta\omega(j))^2 + \gamma^2(j)/4\right] \tag{3.16}$$

with nonresonant factor $A = \pi\sigma_{res}\Gamma^2(\Delta\omega^2 + \Gamma^2/4)/(\Delta\omega + \delta\omega)^2$ and with shifts of effective levels and width:

$$\delta\omega = (j\sigma_{res}\Gamma)^{1/2},$$
$$\gamma^2(j)/4 = \frac{\Delta\omega - \delta\omega}{\Delta\omega + \delta\omega}\cdot\frac{\Gamma^2}{2} + \frac{\Gamma^4}{16(\Delta\omega + \delta\omega)^2}. \tag{3.17}$$

These expressions demonstrate that reemission processes lead to the renormalization of parameters of electron levels. At the strict resonance $\Delta\omega = \delta\omega$, $\gamma_{res} = \Gamma^2/4\delta\omega$ and $\delta\omega_{res} = (j\sigma_{res}\Gamma^2)^{1/3}$, which show, in comparison with (3.13), that the functional forms are changing directly at the resonance.

Now we will consider processes of higher field intensity, at $j > j_0$.

As it can be showing, each (nonresonant) act of absorption-reemission adds to (3.8) the multiplier

$$\frac{j}{j^{(1)}} = \frac{\pi c^2 j}{\omega^2}\frac{\Gamma}{(\omega + \omega_0)^2 + \Gamma^2/4} \approx \frac{j}{j_0}\frac{\Gamma^2}{16\omega_0^2}. \tag{3.18}$$

So, if between two resonant transitions appears q nonresonant transitions of the same frequency, the radius of convergence will be increased on the factor of (3.18) in the degree q. Therefore it seems that the convergence of this new reaction rate is possible if the expression (3.18) is not bigger than one, i.e. the radius of convergence with taken into account nonresonant scatterings would be

$$j_1 = j_0(4\omega_0/\Gamma)^2. \tag{3.19}$$

This magnitude is so big (in the optical range $j_1 \sim 10^{15} j_0$) that if only these processes played a role, stimulated radiation would be possible at all practically achievable intensities.

But on the other hand an intensity of irradiation should be lower than the threshold of harmonics generation. At the absence of any additional levels of system, harmonics can arise in the non-resonant way if the distance between consistently suitable quanta is lesser $c\tau_1$, i.e. if

$$j \geq j_{harm} = 1/\tau_1\sigma_{tot}, \tag{3.20}$$

where $\sigma_{tot} = 4\pi c r_0/\omega_0$. As the magnitude of (3.20) is usually much lesser (3.19), it represent natural upper threshold of stimulated radiation connected with opening of a new channel, of HHG.

Notice, that the analogical calculation of stimulated absorption, i.e. of (3.3) with $N \longleftrightarrow N + 1$, shows that corresponding partial rates

$$\frac{R(n \to n+1)}{R(n+1 \to n)} = \frac{n+1}{n}, \tag{3.21}$$

i.e. the validity of the Einstein solution (3.2) of the thermodynamic Diophantine equation.

4 Processes of Higher Harmonics Generation

Generation of higher harmonics, especially at single electrons, can be considered by an analogy with a multiphoton Compton scattering and therefore some problems of its kinematics can be directly transferring from the one-photon processes description. But the dependence on densities of photon fluxes represents some new problems.

Consideration of HHG processes is usually carried out for different energies within the frames of differing representations. For harmonics with energy smaller the ionization threshold it is usually suggested that the electron, which has virtually absorbed some photons, is transferring downwards in the same atom with emission of the saved up energy as one photon. However the HHG of significant big energies is usually considered via three steps: (1) tunnel or barrier-suppression ionization, (2) acceleration by the laser field and redirection to the ion, and (3) recombination with the ion. The excess energy of the recombination process is released as short wavelength radiation (e.g. [1, 44]). The HHGs at the accelerator energy are also investigated [8].

As we shall show, such complications are not necessary: HHG at all energies can be considered as a process when virtually excited electron does not leave the atom, and simply, as well as in the case of weak excitation, comes back onto the place. Underline that the developed approach does not conduct to spectral expansion of the higher harmonics, inevitable at three-stages processes.

However for harmonics of very high numbers and for an explanation of so-called plateau in the harmonics spectrum it is more convenient to use the statistical approach in the frame of QED. Therewith, it appears that there is original rather short-term effect of an equipartition of energy on "degrees of freedom" of electronic gas, which are represented as the number of virtually seized photons.

For comparatively low intensities of laser flux ($J \ll 10^{13}$ W/cm^2), at which atomic levels have significant role, the calculations can be proceeding in accordance to general rules described above. So for the elementary act of frequency doubling (really such process is possible under special conditions in media only):

$$2\gamma(\omega) + e_B \to \gamma(2\omega) + e_B \tag{4.1}$$

we can formally write:

$$R_{1,2} \simeq \frac{2j_f+1}{4(2j_i+1)^2}\frac{\pi^2 j^2}{9\omega^4}\frac{\Gamma^2}{(\omega-\omega_0)^2+\Gamma^2/4}\frac{\Gamma}{(\omega-2\omega_0)^2+\Gamma^2/4} \sim Q\Gamma(j/j_0^{(2\to1)})^2, \tag{4.2}$$

where Q include all multipole factors and the critical flux density

$$j_0^{(2\to1)} = \sqrt{1/\sigma(\omega)\tau(\omega)\cdot\sigma(\omega)\tau(2\omega)} \tag{4.3}$$

is the geometrical average over one-photon densities.

Reaction rates for generation of higher harmonics on isolated scatterer can be written analogously:

$$R_{1,n} \sim Q\Gamma j^n/j_0^{(n\to1)} \simeq Q\Gamma j^n\sigma^n(\omega)\prod\tau(\omega-k\omega_0). \tag{4.4}$$

This approach can be essentially simplified. As a matter of fact from QED is required only the concept of time of delay and the formations of final states applying to statistical consideration of processes. Simultaneously such consideration reveals some features of MPPs.

Let's consider a set of free electrons which is irradiated with a monochromatic photon flux $J = \hbar\omega j$ of high intensity. Process of elastic scattering occurs as absorption of a photon and its emission through time of delay τ. But if the density of photons flux is such that during this time the scattering of second photon on the same electron takes place, this electron will save up onto a time duration $\tau/2$ double energy, which it will let out, with greater probability, as one photon of the double frequency. Thus, the electron can save up energy $E_K = K\hbar\omega$, $K = 1, 3, 5, \ldots$, sufficient for HHG, and corresponding momentum, the oddness of K guarantees the parity conservation. This process goes by one stage, can be described by the single Feynman graph and there is not necessity for splitting it onto subprocesses of various nature.

According to the uncertainty principle if the electron absorbs K photons of energy $\hbar\omega$, it can keep their energy during the time $T'_K = 1/2K\omega$. The theory of temporal functions overviewed in the Sect. 1 gives twice bigger value till the end of particle formation (its "dressing"): $T''_K = 1/K\omega$. Hence we can estimate the duration of holding time for accumulated energy as

$$T_K = 1/\eta K\omega \quad (1 \le \eta \le 2). \tag{4.5}$$

Over this time N_K photons are transferring through the maximally possible cross-section $S_0 = \alpha\pi\lambda^2$ of e-γ interaction (α is the fine structure constant):

$$N_K = j\xi S_0 T_K, \tag{4.6}$$

where ξ characterizes a competition in photons capture with other sufficiently closely located scatterers (it will be considered below and can be omitted at consideration of rarefied gas).

The probability of each virtual absorption (capture) is proportional at considered frequencies and at absence of resonances to the total cross-section of the single photon scattering $\sigma_{tot} = 4\pi c r_0/\omega$ and can be represented as

$$p \simeq \sigma_{tot}/S_0 = \frac{2}{\pi}\lambda_C/\lambda, \tag{4.7}$$

λ_C is the Compton wave length, in considered processes $\lambda \gg \lambda_C$.

The probability of virtual absorption of K photons from among N_K quanta passing through the maximal possible interaction cross-section, is expressed by the binomial distribution (e.g. [14]; for its approximations we use, actually, the conservation laws only):

$$B_K(N_K) = \binom{N_K}{K} p^K (1-p)^{N_K}. \tag{4.8}$$

This distribution corresponds to a fast decreasing of intensity of lower frequencies harmonics, but for big $N_K \gg 1$ and with $pN_K \ll 1$ this decreasing is sharply slowing down. So the distribution (4.8) can be approximated at these conditions by the Poisson distribution with the mean value of virtually absorbed photons forming the K harmonic:

$$\overline{n}_K = pN_K = j\sigma_{tot}T_K = \frac{J}{\hbar\omega}\frac{4\pi c r_0}{\eta K\omega^2}. \tag{4.9}$$

It means that the mean energy of this harmonic

$$\overline{E}_K = K\hbar\omega\overline{n}_K = J\frac{4\pi c r_0}{\eta\omega^2} = J\frac{\sigma_{tot}}{\eta\omega}, \tag{4.10}$$

i.e. this mean energy does not depend on the harmonics number in the considered part of the HHG spectrum and forms the observed plateau. (To this value can be added, of course, $I/\hbar\omega$, where I is the ionization potential.)

Hence each harmonic in this region of spectrum can be considered as the "degree of freedom" of photonic gas with the equipartition law of their description when the interaction between scatterers is negligible. Moreover (4.10) does not contain $\hbar$ and therefore appears as the classical magnitude in the concordance with the classical, generally speaking, notion of degrees of freedom.

This expression can be rewritten as $\overline{E}_K = (4/\eta)U$, where U is the ponderomotive potential. Note, that in so called "simple man's theory" [10, 24]; this energy is introduced for description of the experimental data as $3.17U$. It means that the coefficient η is precisely in the limits of (4.5), $\eta \in [1, 2]$, and that, consequently, our result qualitatively, at the least, conforms to the experimental data and therefore further numerical comparisons are not required.

The higher frequencies tail of (4.8) at big K and $\overline{n}_K$, but with $K/\overline{n}_K \to 1$, can be approximated by the Gaussian:

$$B_K(N_K) \to (2\pi\overline{n}_K)^{-1/2}\exp(-(K-\overline{n}_K)^2/2\overline{n}_K). \tag{4.11}$$

It evidently corresponds to the observed decreases of energy of harmonics with the rising of $K \to N_K$ after the plateau.

Let's consider now the low-frequencies tail of harmonics spectra. With the lowering of kept energy the duration of their keeping will increase, and therefore possibilities of their partial transferring to another scatterers will also increase; it will lead to a distortion of the harmonics' profile, to formation of quasi continuous spectrum.

This transferring is effective for such accumulated energy only, for which cT_K is bigger $\rho_e^{-1/3}$, the mean distance between scatterers. Therefore we can expect that the clear harmonics comb begin only with K bigger

$$K_{\min} \sim (c/\eta\omega)\rho_e^{1/3}. \tag{4.12}$$

Hence we had finished, in principle, the description of the main parts of HHG spectra for rarefied scatterers set. For consideration of HHG processes in more dense media a coefficient ξ in (4.6) must be estimated.

The simplest case is a rarefied medium of the scatterers' density ρ_e, when the free path length of photon $\ell = 1/\rho_e\sigma_{tot}$ is bigger the target depth L, then the coefficient $\xi = L/\ell$.

For more dense targets, but with $L \le \ell$ and when the photon wavelength $\lambda \gg \rho_e^{-1/3}$, the sufficiently thin target gas layer can be subdivided onto "interaction pipes" of cross sections $S_1 = \pi\rho_e^{-2/3}$, so it can be assumed that each photon can interact in each "pipe" with a single scatterer only. If we shall accept, for simplicity, that $\rho_e = Z\rho$, ρ is the atomic density, Z is a number of electrons taking part in the process (note that at sufficient energies all atomic electrons, not only valent ones, can be involved in the reaction), then into the expression (4.6) would be inserted

$$\xi = (Z\rho)^{-2/3}/\alpha\lambda^2, \tag{4.13}$$

and the probability of single photon virtual capture will be now, instead of (4.7), determined as $p_1 = \sigma_{tot}/S_1$.

According to (4.6) the maximal photons quantity, that are passing through the "interaction pipe" and can be virtually captured by the scatterer, will be equal to

$$N_{\max} = jS_1 T_{\min} \simeq jS_1/\eta\omega N_{\max}, \tag{4.14}$$

i.e. the highest harmonic of plateau in the HHG process will be of the order

$$N_{\max} \simeq (\pi/\eta)^{1/2}(J/\hbar\omega^2)^{1/2}(Z\rho)^{-1/3}. \tag{4.15}$$

Hence the numbers of higher harmonics are proportional to the strength of laser field, instead of its power. Such dependence seems physically natural for the common classical representations that the energy of electron is determined by its acceleration in the laser field.

The dependence $Z^{-1/3}$ for the highest achievable harmonic (4.15) can be comparable with some experimental data. In the article of Peatross and Meyerhofer [32], the highest numbers of harmonics formed by the various gas targets are concerned as $Ar : Kr : Xe = 1 : 0.85 : 0.71$, the expression (4.15) results in ratio $1 : 0.79 : 0.69$. In the article of Chang et al. [9], the ratio of the highest harmonics numbers on targets $He : Ne = 0.58$, our formula gives for them 0.52. According to Corcum [10], such relation was measured for gases $Ne : Ar \geq 0.75$ (uncertainty is connected to alterations of gas density in this research), the relation (4.15) gives 0.75.

With dependence on frequency the situation is more complicated: the estimation (4.15) results in dependence $1/\omega$, but for the highest harmonics and in sufficiently dense medium only. The similar estimation for rarefied gases (4.9) leads to $1/\omega^2$, which is closer to the "simpleman's theory", where $N_{\max} \sim I/\hbar\omega + a\sigma J/4\hbar\omega^2$, with $a \sim 2 \div 3$. The dependence of $N_{\max}$ on wavelength with equal other conditions was measured by Shan and Chang [45], at $J = 4 \cdot 10^{14}$ W/cm^2 for wavelengths 0.8 and 1.51 mcm: the cut off of harmonics set occurs, correspondingly, on energies of 64 eV and close to 160 eV that corresponds to $\lambda^{1.8}$ and is close enough to square-law dependence in (4.10), but not to (4.15). This observation can be interpreted as the obvious minimization of the role of interaction with neighbor scatterers at higher frequencies with their shorter durations for such interactions, but evidently requires further investigation.

For numerical estimations, however, the more precise dynamical models are needed. Nevertheless, for a characteristic set of used parameters: $J = 10^{14}$ W/cm^2, $\hbar\omega = 1$ eV, $Z = 18$, $\rho = 10^{18}$ cm^{-3}, and at assumption that all processes are channeled only in the HHG we receive that $N_{\max} \geq 200$. This estimation does not seriously contradict observed values, especially since even the account of polarization of a laser beam and its angular divergence can reduce this value. (Notice that for the mean number of accumulated photons the expression (4.9) gives more adequate answers.)

It must be underlined that all above is conditioned by a shortness of laser pulse. If its duration would be longer, it can stimulate the partial exchange of surplus energy between electrons and accelerate the leveling of harmonics comb. It is not excluded that just such reasons can explain the phenomena, observed in some experiments [7], when the intensity of harmonics at the light impulse in 7 fs is approximately on the 0.5 orders above than at the impulse in 30 fs. Note that inasmuch as at the initial stage of dissipation its speed corresponds to the plasma frequency of electronic components $\nu_p = (e^2\rho/\pi m)^{1/2}$, the condition of smallness of thermalization can be estimated through smallness of the laser pulse duration in comparison with $1/\nu_p$.

The described picture can be evidently generalized on the case of two or more laser beams of different frequencies: the spectrum of harmonics will be more complex, inasmuch as, in the accordance with the composition law of binomial distributions, the new and more complicate set of "the degrees of freedom" can appear, but it does not change the general approach and observed picture.

The processes of summation of high harmonics quanta (including quanta of different frequencies) with each other are also possible. Probably, the mechanism of some harmonics amplification in the research of Bartels et al. [3], where the HHG was spent out in the long capillary with gas, is just those that increased the probability of virtual absorption and subsequent addition of previously formed quanta of higher frequencies.

At processes of HHG in the unidirectional flux all consideration is facilitated by the concordance of energy and momenta of harmonics quanta. In other cases, at the ATI and even in the crossed laser beams, the accumulation of additional momentum via interaction with parent ions or with neighbors is needed; it can accelerate the thermalization effects and widening of single harmonics (the extreme case of crossed light beams are processes in the opposite directed laser beams [47]). The kinematics of such processes is evident and can be omitted.

We did not consider above the dependence of sequential photons capture on their polarization. As the virtual absorption of the first photon polarizes electron, it becomes necessary to take into account that the probability of following photon absorption would be proportional to probabilities of polarized photons scattering on polarized electrons. This circumstance, it is possible to think, changes the intensity of HHG at division of focuses of laser fluxes with different linear polarizations and at circular polarization in comparison with linear. But it also does not alter basically the described picture.

The revealed features can be considered as the extending of the Rayleigh-Jeans theory of the equilibrium low-frequency thermal radiation to the rapid energies reallocation at power laser flux interaction with electrons set. At the considered phenomenon the role of black body fulfill scatterers with opportunities of only discrete and equidistant, on the frequencies $K\omega$, virtual energy accumulations. Note that the rescattering phenomena as the cause of plateau formation (e.g. [17], and references therein) and our approach lead to different temporal dependences, but we did not see yet sufficient experimental data for the choice between them.

It can be interesting to note that in the articles of Jeans [19, 20], where for the first time the fluctuations of "ether" in the resonator and their harmonics had been considered, only the equality of possible energies, without their equating to $\kappa T/2$, was accepted.

Some other peculiarities of the MPPs consist in the dependence of maximal number of these virtual degrees of freedom on the density of photon gas, on the target parameters and on the specific short-term condition of "thermodynamic balance". It is remarkable that in an accord with the classical definition of degrees of freedom their energies are classical magnitudes.

5 Multiphoton Ionization: Problem of Momentum

Consideration of processes of multiphoton ionization (MPIs) can be beginning with comparisons of reaction rates of one-photon processes and HHG on single scatterers with equal final energies and the same energy of an intermediate state:

one-photon processes with rates $R_{el} = j\sigma_{el}$ and $R_{ion} = j\sigma_{ion}$,

$$\gamma(\Omega) + A \to \gamma(\Omega) + A, \tag{5.1}$$

$$\gamma(\Omega) + A \to e(mv^2/2) + A^+, \tag{5.2}$$

and multiphoton processes with rates R_{HHG} and R_{MPI},

$$n\gamma(\omega) + A \to \gamma(\Omega) + A, \tag{5.3}$$

$$n\gamma(\omega) + A \to e(mv^2/2) + A^+, \tag{5.4}$$

where $n\hbar\omega = \hbar\Omega = I_0 + mv^2/2$, I_0 is the potential of ionization.

Their rates must be connected by the ratio:

$$R_{el}/R_{ion} \simeq R_{HHG}/R_{MPI}, \tag{5.5}$$

it follows unitarity of S-matrix since all four processes have almost identical (last for (5.3–5.4) graphs) intermediate states of an electron virtually absorbing energy $\hbar\Omega$ with duration of this stage $1/2\hbar\Omega$. The ratio can be comparable with the Fermi-Watson theorem for π-mesons photoproduction [16].

The rates of one-photon processes are well known and if the HHG process was examined, the rate of MPI can be described via them. If this ratio is not executed, it would mean that MPI goes, completely or in part, as a multi-stage process, the initial formation of some high harmonics and subsequent ionization by one or even some different photons of high harmonics.

For the MPP with such (last) intermediate state as the combination of (5.3) and (5.2), the reaction rate can be estimated:

$$R^{(1)}_{MPI} \sim R_{HHG} T_{HHG} R_{ion}, \tag{5.5$'$}$$

where T_{HHG} is the duration of (5.3) that via the uncertainty principle can be approximated as $1/2\Omega$. Hence, with taking into account (5.5) and $R_{el} = j\sigma_{el}$, where $\sigma_{el} \approx \sigma_T$, the classical Thomson cross-section,

$$R^{(1)}_{MPI} \sim \frac{1}{2\Omega} R_{HHG} R_{ion} = \frac{\sigma_T}{2N\omega} j R_{MPI} \tag{5.5$''$}$$

and it can be generalized on more number of intermediate states.

Let's consider now the direct calculation of R_{MPI}. It can be approximately represented as

$$R_{MPI} \to R_{HHG}(E) D(E, \mathbf{p}), \tag{5.6}$$

where $E = \hbar\Omega - I$ is the kinetic energy of photoelectron, $p = \sqrt{2mE}$ is its momentum, $D(E, \mathbf{p})$ is the probability of momentum accumulation via interaction with atomic reminder and so on.

The problem of momentum accumulation by freed electron had been the item of discussing from the very beginning of quantum electrodynamics [6], and even earlier, in the theory of one-photon photoeffect (e.g., [4, 48], the modern statement Amusia [2]) and in the theory of molecules photo-dissociation. As can be represented, the most direct decision of this problem can be achieved by the Landau method advanced in the theory of predissociation ([25]; more detailed consideration in [26]); therefore the state of an electron that receives enough energy, but the corresponding momentum has not accumulated up yet, can be named the pre-ionization state.

Let's try to estimate $D(E, \mathbf{p})$ in (5.6) by the Landau method.

Let ψ_b and ψ_f describe states of an electron in discrete and continuous spectra,

$$(\mathbf{H}_0 + \mathbf{V})\psi_{b,f} = U_{b,f}\psi_{b,f}. \tag{5.7}$$

If $\psi_{0,b}$ and $\psi_{0,f}$ are the self-functions, and U_{0b} and U_{0f} are the self-values of the Hamiltonian $\mathbf{H}_0$, it is possible to search for wave function as the superposition $\psi = 441\psi_{0,b} + 441\psi_{0,f}$. Multiplying (5.7) at the left serially on the functions conjugated to $\psi_{0,b}$ and $\psi_{0,f}$, we receive system of two homogeneous equations and their condition of compatibility results in the expression:

$$U_{b,f}(\mathbf{r}) \equiv \overline{U} \pm \Delta U = (U_{0b} + U_{0f)}/2 \pm [(U_{0b} - U_{0f})^2 + 4U^2]^{1/2}, \tag{5.8}$$

where $U(\mathbf{r})$ is the matrix element of the interaction operator $\mathbf{V}$, and all magnitudes are functions of $\mathbf{r}$.

The start of photoelectron is possible from the point, in which $\Delta U = 0$. Near to this point the difference $(U_{0b} - U_{0f})$ as function of $\xi = |\mathbf{r} - \mathbf{r}_0|$ can be approximated as $U_{0b} - U_{0f} \approx (\Im_f - \Im_b)\xi$, where $\Im = -(dU/d\mathbf{r})$ is the force of interaction between an electron and atomic reminder. As this process can be considered as nonrelativistic, it is possible to accept that $\xi = vt$. Then the equality $\Delta U = 0$ shows that the transition occurs during (pure imaginary) moment of time

$$t_0^{(\pm)} = \pm 2i\,|U|/v\left|\Im_f - \Im_b\right| \equiv \pm i\tau(\mathbf{p}), \tag{5.9}$$

its completely imaginary form corresponds to the duration of final state formation τ_2 described in [39].

If $\Im_f \ll \Im_b$ in (5.9) also, it is possible to assume that $2\,|U|/v$ is about the momentum, which has been accumulated up by an electron at a starting from atom; then $\tau(\mathbf{p})$, in the full conformity with the Second Law, shows the duration of time necessary for accumulation of the momentum corresponding to kinetic energy of electron:

$$\tau_n(\mathbf{p}) = \Delta p_n/\Im_b, \tag{5.10}$$

where $\Delta p_n \approx [2m(\hbar\Omega - E_n)]^{1/2}$ at ionization within the state with the main quantum number n. Thus it can be assumed that $\Im_b \sim E_n/a_n = \frac{1}{2}(Z\alpha)^2 \cdot Ze^2/(n^2\lambda_C)^2$, i.e. the duration of time, necessary for momentum accumulation, sharply increases with transition to highest levels that results in a relative stabilization of the Rydberg levels concerning opportunity of MPI.

Accepting to $\Delta p_n \sim [2m\hbar\Omega]^{1/2}$ it is received that

$$\tau_n(\mathbf{p}) \approx 1.3 \cdot 10^{-17} n^4 Z^{-3} (\hbar\Omega)^{1/2}, \tag{5.10$'$}$$

where $\hbar\Omega$ is expressed in the eV's.

This magnitude is the major parameter since it is obvious that for realization of considered reaction the durations of momentum and energy accumulations should be coordinated.

Now it is possible to search for the solution of the Schrödinger temporal equation, needed for their coordination:

$$i\hbar\partial\Psi/\partial t = [\mathbf{H}_0(t) + \mathbf{V}(t)]\Psi, \tag{5.11}$$

in the form

$$\Psi = b(t)\psi_b + f(t)\psi_f \tag{5.11$'$}$$

with the initial conditions $b(-\infty)=1$, $f(-\infty)=0$, such that $|f(\infty)|^2$ is the probability of electronic transition in a point $\mathbf{r}=\mathbf{r}_0$ into the state ψ_f, and $|b(\infty)|^2 = 1-|f(\infty)|^2$ is the probability of electron that further stays in the atom.

According to the quasiclassical Landau method for the solution of such equations, it is necessary to proceed in the factor

$$b(t)=\exp\left\{-(i/\hbar)\int_{-\infty}^{t} U(t)dt\right\} \tag{5.12}$$

at big $t<0$ into the complex plane with detour the point $t_0^{(+)}$ from above. Thus $b(t)$ passes in $f(t)$ that conducts to the expression:

$$t|f(\infty)|^2 = \exp\left\{-(i/\hbar)\int_{-\infty}^{i\tau(p)} \Delta U(t)dt\right\} \longrightarrow \exp\{-\pi\Omega\tau(p)\}, \tag{5.13}$$

where $t'=0$ is accepted.

It is natural to assume, in accordance to the uncertainty principle, that duration of holding of virtually absorbed energy of N photons by an electron, i.e. duration of formation of a free electron state with kinetic energy $\hbar\Omega \simeq p^2/2m$, is about $\tau_2 = 1/2\Omega$. Hence, the parameter exhibitors in the expression (5.13) is determined by the ratio of durations necessary for accumulation of mutually corresponding momentum and energy of the taking off electron. According to (5.10′) this key parameter of the theory can be appreciated as

$$\zeta \equiv \tau_n(\mathbf{p})/\tau_2 \approx 0.04 n^4 Z^{-3}(\hbar\Omega)^{3/2}. \tag{5.14}$$

The full probability of pre-ionization can be determined according to the Landau-Zener definition of probability of predissociation as

$$P_{LZ} = 2|f(\infty)|^2(1-|f(\infty)|^2) \equiv 2\exp(-\pi\zeta/2)[1-\exp(-\pi\zeta/2)]. \tag{5.15}$$

The expression (5.15) has the flat maximum equals $\frac{1}{2}$ at

$$\zeta = (2/\pi)\ln 2 \approx 0.44, \tag{5.16}$$

weakly varies at $\tau(\mathbf{p})/\tau_2 \sim 0.2 \div 0.8$ in an interval about $\zeta = 0.4 \div 0.44$ and decreases on the order at $\zeta < 0.1$ and $\zeta > 3$. From last condition it follows that the efficiency of the direct MPI should fall down essentially at

$$\hbar\Omega_{max} \geq 75 n^{-4} Z^3 \text{ [eV]}. \tag{5.17}$$

Thus, MPIs at big energies should, basically, pass through stages of accumulation of energy, virtually corresponding to HHGs, and the subsequent one-quantum photoeffect. Therefore the spectra of MPIs have the plateau analogically to HHGs processes. Notice, that the stabilization of the Rydberg levels at MPI and necessity of transition to heavier atoms for increasing the output of photoelectrons also follows (5.17).

Thus, this factor shows definite reduction of probability of ionization with growth of energy in comparison with probability of HHG. However, speed of this falling off is proportional N^{-1}, i.e. obviously is slower than at three stages MPI processes: so it can be concluded that the basic role in MPI in the field of low frequencies executes direct liberation of electron that accumulated energy of absorbed photons in atom.

Note that processes of electrons acceleration by a powerful laser field

$$N\gamma(\omega) + e_F \to e_F(mv^2/2) + \gamma(\Omega_1) \tag{5.18}$$

at $mv^2/2 = \hbar(N\omega - \Omega_1)$ can be considered as the multiphoton Compton effect. Energy absorption at this process goes as described above, the necessary momentum is got due to direct interaction with the field considered through Lorentz's forces. Therefore consideration of this process can be omitted here.

6 Radiative Phase Transitions of the First Kind in Two-Level Model

Let us consider such simple model of media: two phases, e.g. gas and solid or liquid are presented as a system of two levels separated by the latent energy Λ/N_A per particle at constant temperature, Λ is the molar latent heat, N_A is the Avogadro number. Then the phase transitions can be considered by some analogy with the Sect. 3: photons generated in the course of transition, i.e. at descent of each particle onto the condensate level, can be termalized in media or, at some conditions, can be emitted as real photons. Moreover, an existence of induced radiation is not excluded in such picture and it can lead, in principle, to stimulated phase transitions into definite condensate state [34].

The main difference with calculations above consists in the kind of electron propagator. Instead of (2.3) here must be taken the temperature-temporal propagator of bound electron in media (e.g. [27]):

$$G_{\nu,\lambda}(x_1, x_2; T, \mu) = -i \sum_{N,n} e^{\beta(\Omega+\mu N - E_n)} \left\langle E_n, N \left| \left\{ \psi_\nu(x_1), \overline{\psi}_\lambda(x_2) \right\}_+ \right| E_n, N \right\rangle, \tag{6.1}$$

where ψ_λ is the wave function of atomic electron, n are numbers of levels with halfwidths Γ_n; at $T = 0$ this expression turn into the Low propagator (2.3).

For non-ferromagnetic media the Fourier transformation of (6.1) over time takes the form:

$$\begin{aligned} G_{\nu,\lambda}(\mathbf{r}_1, \mathbf{r}_2; \omega) = {} & -(2\pi)^3 \sum_{N,n,m} e^{\beta(\Omega+\mu N - E_n)} \\ & \times \frac{1}{2s+1} \sum_\alpha (\psi_\nu(\mathbf{r}_1))_{n,m}, (\overline{\psi}_\lambda(\mathbf{r}_2))_{n,m} \frac{1 \pm \exp(-\beta\omega_{n,m})}{\omega_{n,m} - \omega - i\Gamma_n}. \end{aligned} \tag{6.2}$$

Direct calculations of bound electron-photon interactions at $T \neq 0$ are not necessary because in all respects they differ only by the extra factor

$$\Delta_n = (1 + \exp(-\beta\omega_0)) \exp[\beta(\Omega + \mu N - E_n)] \tag{6.3}$$

for each electron line from the previous ones. The number of photons emitted at transition of single particle is not specialized here, if the symmetry of each particle is not changed at the course of transition, the most probable will be emission of two photons in the ${}_1S^0$ state.

Hence the unique difference of such relations for phase transitions from the above formulae of stimulated emission reaction rates consists in a replacement of the flux density threshold values:

$$j_0 \to j_0^{(phase)} = j_0/\Delta_1\Delta_2 \simeq (\sigma\tau\Delta_1\Delta_2)^{-1}. \tag{6.4}$$

The existence of pole (6.4) in every infinite virial series for bound electron-photon flux interactions leads to the following picture: when the temperature is increasing and the density of thermal radiation in frequency band near resonance becomes equal to (6.4), then the phase transition takes place: a saturation of one type of interactions and the including of following channel of interaction with another features (condensate turns into gas phase and so on).

Let's consider the free energy jump determining the saturation of some e-γ interaction channels. From (6.3) and (6.4) follows

$$\Omega_n = \frac{1}{2\beta}\{-\mu N - E_n - \ln\tau_n - \ln[2\sigma j(1+\cosh\beta\omega_0)]\}. \tag{6.5}$$

At transition from one type of interaction to another the jump of energy is of order of

$$\Delta\Omega_n = \Omega_{n+1} - \Omega_n \approx -(1/2\beta)\ln(\tau_1/\tau_2) \sim -(1/\beta)\ln(\Gamma/2\omega_0). \tag{6.6}$$

This relation can be rewritten via the molar latent heat and with the universal gas constant R_g as

$$\Lambda/T \sim R_g \ln(2\omega_0/\Gamma). \tag{6.7}$$

In the infrared range, where characteristic emissions of latent heat may be expected, it can be proposed that $\omega_0/\Gamma \sim 10^4$. It leads to $\Lambda/T \sim 24$ cal/mol·K that just corresponds to the well-known empirical Trouton rule established in 1884 for many substances (non-polar or weakly polar): latent heat and temperature of boiling are connected at normal pressure by the relation of order $\Lambda_b/T_b \sim 21$ cal/mole·K. This rule is often used for physicochemical estimations (cf. [38, 41]). However, till now this rule has not been substantiated theoretically; its sense and significance remained unknown. The estimation (6.7) can be indubitably concretized for different transitions and substances.

7 Multiphoton Processes in Thermal Radiation Fields: Astrophysics and Cosmology

The density of photons in the Rayleigh-Jeans part of the thermal spectrum is so great that, with presence of suitable scatterers, multiphoton processes on them inevitably should be executed. Such processes should result in distortions of the Planck spectrum with pumping of energy on higher frequencies, and also should deform the low frequencies part of linear spectrum. Therefore the account of these processes can have essential value at the analysis of bright flashes, at discussions of problems of explosions, etc.

Thus, the astrophysics should be the second, after laser physics, object of research of MPPs [36].

The main difference with all above is evident. The density of photons flux in (2.9) and (3.6) must be represented via the Planck distribution:

$$f(t,\mathbf{k}) = 2\left(\frac{r_0}{r}\right)^2 \frac{1}{\exp(\hbar\omega/kT)-1}, \tag{7.1}$$

where r_0 is the radius of source and r is the distance of scatterer at which MPPs are executed.

For the induced radiation in two-level system it leads with taking into account the δ-form of propagators to such partial rates:

$$R_{n+1,n} \sim n\Gamma\left[2^5(r/r_0)^2(\exp(\hbar\omega/kT)-1)\right]^{-n} \tag{7.2}$$

and correspondingly to the complete reaction rate $R = \sum R_{n+1,n}$. With these expressions the temperature threshold of stimulated radiation follows:

$$\lambda T \geq 2\pi(\hbar c/k)\ln[1 + 2^{-5}(r_0/r)^2]. \tag{7.3}$$

At $r \sim 2r_0$ it leads to $\lambda T \geq 100$ cm·K, i.e. the inducing radiation is possible very far from the maximum of spectrum ($\lambda T_{\max} \sim 0.29$ cm·K) and comes in the visual region at $T \geq 10^6 \div 10^7$ K, e.g. at flares of nova stars, etc. But in the IR range at some flares on the Sun or in its microwave part these phenomena can lead to the lines narrowing and shifts.

Temperature thresholds of HHG and MPI can be estimated analogically.

The simplest process of frequencies addition can be represented by such chain with resonant atomic level (ω_1, Γ_1):

$$\gamma(\omega_1) + e \rightarrow e^*(\omega_1), \qquad \gamma(\omega) + e^*(\omega_1) \rightarrow e + \gamma(\omega + \omega_1), \tag{7.4}$$

if it absorbs second quantum earlier of time $2/\Gamma_1$, then it emits summary frequency. With growth of probability of such processes the line begins to widen and at sufficient intensity can cease to be observable.

The threshold of second process (7.4), i.e. its observability, is of the order

$$I(\omega) = (r_0/r)^2\omega^3 kT/6\pi^2c^2 > \hbar\omega/\tau_1\sigma(e), \tag{7.5}$$

$I(\omega)$ is the density of radiation flux in the Rayleigh-Jeans interval $(0, \omega)$.

With MPPs leading to levels widening can compete processes of stimulated radiation $\gamma(\omega_1) + e^* \rightarrow e + 2\gamma(\omega_1)$, which would not be realizable if

$$I(\omega_1 + \Gamma) - I(\omega_1 - \Gamma) \approx (r_0/r)^2\omega_1\Gamma kT/\pi^2c^2 < 2\hbar\omega_1/\sigma(\omega_1)\Gamma. \tag{7.6}$$

So it can be shown that the widening of spectral line has place in an interval:

$$\omega_1/\Gamma < kT/\hbar\omega_1 < (\omega_1/\Gamma)^2. \tag{7.7}$$

In the far IR and microwave regions $\omega/\Gamma \sim 10^2 \div 10^3$ and at $T \geq 6 \cdot 10^3$ K lines with $\lambda > 10^{-2}$ cm can be widened (the lower estimation).

This effect should be observable in spectra of objects, in which the transformed radiation passes through so thin layer of gas that has not time for thermalization, e.g. at early stages of supernova explosions. Therefore this effect can explain absence of spectral lines at early stages of flash and that the display of lines begins with high-frequency ones. Besides it can explain an origin of a step in the premaximal spectrum: it is caused by multiphoton transferring of energy into high-frequencies (hence calculation of energy of flash by a visual part of spectrum can give underestimated values). Note, for example, that exactly such step was observed at flash of SN1987A [31].

The opposed processes of pumping lines from the continuum begin with lower temperatures. So if $\omega' + \omega'' = \omega_1$, the MPP

$$\gamma(\omega') + \gamma(\omega'') + e \rightarrow \gamma(\omega_1) + e \tag{7.8}$$

is characterized by the threshold

$$\lambda' T \geq 10^2\omega'\omega''/\omega_1^2 \text{ [cm} \cdot \text{K]} \tag{7.9}$$

and can be observed as dips around ω_1 in the spectra of hot stars. Thus the temperature defined via lines will be bigger than defined via continuum

These transformations of spectra by multiphoton processes allow to naturally explain, for example, some features of flashes of UV Cet type stars: (a) The UV excess, (b) Rise of blue coloration of spectra, (c) Occurrence of emission lines with higher ionization potentials, (d) The maximum of emission in lines begins later than in the continuum and lines are flashing longer, (e) Lines become rather flat, their seeming width grows till dips in the line centers (e.g. [18]).

Theory of MPPs gives, anyway, a qualitative explanation of all these phenomena. So, the UV excess and the general blue coloration of spectra speak on pumping of energy in a spectrum by MPPs addition of frequencies; occurrence of new resonant lines is the result of multiphoton excitation; delay of radiation maxima in lines, as well as their bigger flattening, is connected to transferring lines energy into continuum at frequencies summation.

Thus, all prominent features of the spectrum of these (comparatively low energy) flashes can be understood within the frame of the hot spot theory at the taking into account the phenomena of MPPs.

The most powerful sources of radiation, the gamma-ray bursts (e.g. [29]), are corresponding to sources, apparently, that are heated up to virial temperatures in the multi-MeV range. Notice that their spectra would be characterized, in particular, by giant intensity of radiation in low-energy parts (visual, UV, etc.) also. But this radiation is absent.

It can be suggested that this feature would be explained, the most likely, by multiphoton processes, by HHGs and multiphoton birth of pairs with transition of radiation energy to characteristic frequencies. These processes can be considered, in particular, as HHG on vacuum loops and, at bigger energy, as their multiphoton disclosing. Therefore we shall describe such processes in the general form for electromagnetic and gravitational fields.

In the field of q-type the existence of virtual vacuum pair of particles of masses m can be continued till the moment

$$\tau_q \sim \hbar/2(2mc^2 - W_q), \tag{7.10}$$

where W_q is the energy of field in an effective volume of pair $V = \lambda_C\ \sigma_T$. During the time (7.10) virtual pair can absorb the energy

$$\hbar\Omega = \tau_q \left(\frac{r_0}{r}\right)^2 \int_0^\infty I(\omega)\sigma(\omega)d\omega \le \tau_q \sigma_T \sigma_{SB} T^4 \left(\frac{r_0}{r}\right)^2 \tag{7.11}$$

and consequently can emit at annihilation act two, at least (for parity conservation), photons with general energy (7.11). (In some sense this effect can be analogical to direct HHG on vacuum pair by intensive laser field: Fedotov and Narozhny [15], and as a pair loop disclosing: Blaschke et al. [5].)

If $\hbar\Omega \ge 2mc^2$, this process can lead to the birth of real pair observed as the direct transition in vacuum:

$$\gamma(\omega_1) + \gamma(\omega_2) + \cdots \to e^- + e^+ + \cdots. \tag{7.12}$$

In the region, where $W_q/2mc^2 = 0.9$, this process must be observable at $T \sim 10^{10}$ K, but close to the horizon of black hole, where $W_q/2mc^2 \to 1$, it can be executed by accumulation of quanta of much lower thermal radiation. Further annihilation of the pair (7.12) would lead to appearing of characteristic frequencies in the high energy part of spectra. Thus the MPPs lead to some broadening of the holes horizon.

In the magnetic field $W_m = (r_0/r)^2 \mathbf{B}^2 V/8\pi$ and it can lead to the pair birth close to pulsars surface. The needed conditions for such effects exist at ultra-high magnetic field neutron stars (magnetars). The needed fields can be, in principle, artificially produced [21]; analogical effect can be induced by intensive laser flux in the atomic Coulomb field, e.g. [23], and references therein.

It is necessary to note the consequences of the MPP theory in cosmology, for the Big-Bang theory. The matter is that for differentiation of epochs of those or other types of particles formation the one-photon processes of pairs birth are usually examined, therefore the corresponding temperatures are suggested as $kT_i = m_i c^2$. But in accordance with (7.10) at $\hbar\Omega = 2m_i c^2$ we receive, that

$$kT_i \approx m_i c^2 \left[\alpha^{-2}(1 - \chi/2c^2)\right]^{1/4}, \tag{7.13}$$

where $\alpha = 1/137$ and χ is the gravitational potential. Hence MPPs essentially shift threshold temperature of hadron and lepton eras of the Universe and extend them.

8 Conclusions

Let us enumerate considered problems and the main results.

1. Consideration of processes with N incident photons ($N \geq 2$) in the frame of QED requires the clarification of transition to the infinite volume of field quantization. The refined procedure leads to appearance of dependence of reaction rate on densities of photons fluxes $j(\omega)$ [quant/cm^{2}s].
2. Calculations of MPPs naturally lead to appearing of the time durations of elementary processes which, as all primary physical magnitudes, must be initially contained in the QED and their ad hoc introduction is not needed.
3. Calculation of MPP rates in laser fields leads to "virial" series over (j/j_0). Characteristic photon flux density is presented as $j_0 = 1/\sigma(\omega)\tau(\omega)$, where $\sigma(\omega)$ and $\tau(\omega)$ are the cross-section and time duration of elementary act of elastic e-γ scattering.
4. Reaction rates are characterized by the unremovable pole at $j = j_0$. This pole correspond to saturating of considered process and/or to opening of new channel of interactions and can be naturally interpreted via magnitudes of durations of scattering and cross-sections of scattering.
5. The HHGs can be interpreted via absorption of many quanta during the time of keeping, in accordance with the uncertainty principle, a sum of their energy by bound or free electron. The emission of all this energy by one photon is the most probable. The HHG on bound electron can completely be executed in the scope of the same atom.
6. The reaction rates of HHGs are expressed via the binomial distribution that leads at cases of big N to the plateau in HHG spectra that is far from both their tails.
7. The MPI process is complicated by the necessity of accumulation of momentum of liberated electron corresponding to energy of virtually absorbed quanta. The computation of this process is executed by the Landau theory of predissociation, that leads to revealing of time duration of momentum accumulation. The reaction rate depends on conformity of both durations.
8. The development of general theory of MPPs and revealing of saturation densities allow to use offered methods and representations not only in the laser physics but for another objects also, more concretely in the thermal radiation fields.
9. As the most interesting phenomenon in the thermal field, the phase transition of the first kind is examined. Two phases, e.g. gas and liquid or solid, can be represented as two levels with radiating transition between them. Our theory predicts emitting of

latent heat as the characteristic radiation with its liberation, which was fixed in some experiments ([46] and references therein). The theory predicts also the possibility of radiative induced phase transitions into one of possible crystal states and so on, but this effect was not yet experimentally investigated.

10. The considered calculations establish the relation between latent heat and the temperature of transition. The relation conforms with the well-known Trouton rule, that was empirically established more than century ago but had not the theoretical substantiation till now. This result releases us from necessity of comparison with experimental data in this item. The used method does not predict the features of such radiation, they are considered in Perel'man and Tatartchenko [41].
11. The most number of phenomena in thermal fields that require theoretical consideration are in astrophysics. The density of photons in the Raylegh-Jeans parts of spectra of bright flares are such big that MPPs of HHG are inevitable. And really some known features of flares can be simply described by this theory. To such phenomena can be referred the basic features of UV Cet spectra, some features of spectral evolution of supernovas and so on. Thus it seems that astrophysics and some problems of cosmology should be considered with due use of the MPPs theory.
12. The developed theory allows to calculate HHGs processes on vacuum fluctuations and disclosing of vacuum loops with generation of particles in gravitational and electromagnetic fields. These phenomena, in particular, displace a little various eras at the Big Bang.

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